

# Majorana fermions in an antiferromagnetic chain in proximity to a superconductor

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We propose a new Majorana fermion platform based on an antiferromagnetically ordered chain of atoms in proximity to a conventional superconductor. This hybrid system can be driven into a topologically non-trivial phase by the combination of a supercurrent flow and a *weak* Zeeman field. Both can be finely tuned providing a platform with enhanced functionality for applications. Remarkably, the electronic spin-polarization of the arising edge Majorana fermions depends on the parity of the number of magnetic moments. The resulting even-odd effect should be measurable and could serve as a signature of the Majorana fermions. We introduce the basic concepts on a model level and confirm them by a microscopic analysis.

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Recently a series of proposals appeared describing how to engineer Majorana fermions (MFs) in hybrid systems based on topological insulators and conventional superconductors (SCs) [1], superfluids [2], non-centrosymmetric SCs [3], and heterostructures of conventional SCs and spin-orbit coupled semiconductors [4–7]. They stimulated a number of experiments [8] striving to identify the emergence of MFs. So far, the most prominent proposal involves an artificial topological SC (TSC) based on a spin-orbit coupled semiconducting nanowire in proximity to a conventional SC. This system can enter the TSC phase in the presence of a sufficiently strong magnetic field.

On the other hand, it has also been shown that neither the magnetic field nor the presence of spin-orbit coupling are indispensable for achieving a TSC [9]. There have been alternative proposals [10–20] involving the combination of conventional SCs and some complex magnetic order. In the majority of these cases engineered helical magnetic fields [21] are assumed, that effectively generate a spin-orbit interaction that can be stronger than the intrinsic one of semiconducting nanowires. Such an inhomogeneous magnetic order can be realized, for instance, by the presence of nano-magnets [11] or by placing magnetic atoms on top of a SC [12–20], where the RKKY-type interaction induces a self-tuned spiral order [16, 17].

In this article we propose a system which combines the advantages of a self-organized magnetic order with additional control fields. The setup consists of a chain of antiferromagnetically (AFM) ordered magnetic moments on top of a conventional SC (Fig. 1). Placing a chain of magnetic atoms on a metal is feasible, as demonstrated by recent scanning tunneling microscope (STM) experiments [22–24]. The orientation of the spins is stabilized by the crystal field of the metallic substrate [22], and the AFM order is stabilized by the RKKY interaction. By imposing a supercurrent flow  $J$  in the superconductor and applying a *weak* in-plane magnetic field  $B$  we can

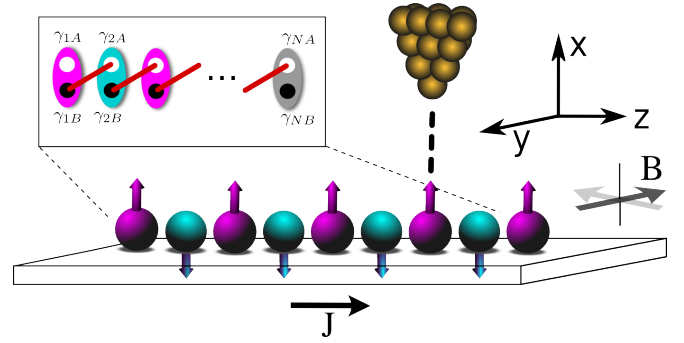


Figure 1: Antiferromagnetic chain on top of a superconductor. A single spin-filtered MF appears at each edge in the simultaneous presence of a weak in-plane Zeeman-field  $B$  ( $yz$ -plane) and a supercurrent flow  $J$  ( $z$ -axis). Inset: Mapping to Kitaev's model with unpaired MFs  $\gamma_{nA,B}$ .

drive the system into a TSC phase with a single MF per edge. The possibility of scanning the topological phase diagram by the two control fields is expected to facilitate the detection of MFs.

In the limit of short superconducting coherence length our microscopically based model for the hybrid system reduces to a *minimal* model combining AFM order, on-site SC, and nearest-neighbor-hopping. In this limit we can map the model onto Kitaev's chain [25] with unpaired MFs which are characterized by an electronic spin-texture [26] that depends on the number parity of the magnetic moments. The resulting *even-odd* effect should be experimentally detectable and could help verifying the emergence of MFs. In the second part of the paper we analyze the model numerically for general parameters and confirm the conclusion derived from the minimal model.

We consider the system illustrated in Fig. 1 consisting of a chain of  $N$  atoms (arranged along the  $z$ -axis) with AFM ordered magnetic moments in  $x$ -direction ( $M$  denotes the corresponding energy scale) and nearest-neighbor hopping  $t$ . The proximity to the superconductor introduces the on-site superconducting order parameter

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$\Delta_n$ . The Hamiltonian is

$$\mathcal{H}^0 = -\frac{1}{2} \sum_{n=1}^N \left[ (-1)^n M \Psi_n^\dagger \tau_z \sigma_x \Psi_n + 2t \Psi_n^\dagger \tau_z \Psi_{n+1} + \Psi_n^\dagger (\Delta_n^{\mathcal{R}} \tau_y \sigma_y + \Delta_n^{\mathcal{S}} \tau_x \sigma_y) \Psi_n \right]. \quad (1)$$

Here  $\sigma$  and  $\tau$  are Pauli matrices in spin and particle-hole spaces, respectively, and  $\Psi_n^\dagger = (\psi_{n\uparrow}^\dagger, \psi_{n\downarrow}^\dagger, \psi_{n\uparrow}, \psi_{n\downarrow})$  is the spin-dependent Gor'kov-Nambu spinor.

In contrast to a spiral magnetic order, considered by a number of authors [11–20], the AFM order alone is not sufficient to generate a transition to a non-trivial TSC phase. However, this can be achieved by the combination of a supercurrent flow along the chain and a perpendicular Zeeman-field in the  $yz$ -plane. The supercurrent introduces a phase gradient in the superconducting order-parameter,  $\Delta_n = \Delta \exp(-iJan)$ , which can be absorbed by a gauge transformation in the fermion fields,  $\Psi_n \rightarrow \exp(-iJan\tau_z/2) \Psi_n$ . This modifies the hopping term in Eq. (1),  $t \rightarrow t \cos(Ja/2)$ , and adds a time-reversal symmetry ( $\mathcal{T}$ ) breaking hopping term  $2it \sin(Ja/2) \Psi_n^\dagger \Psi_{n+1}$ . Together with the magnetic field term, the external perturbations can be written as

$$\mathcal{V} = \frac{1}{2} \sum_n \left[ \Psi_n^\dagger \mu_B B \tau_z \sigma_z \Psi_n + 2it \sin(Ja/2) \Psi_n^\dagger \Psi_{n+1} \right]. \quad (2)$$

To proceed we first discuss the Hamiltonian  $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$  in the limit of an infinite chain with discrete translational invariance. In order to account for the AFM order with wave-vector  $Q = \pi/a$  and lattice constant  $a$  we extend the spinor in momentum space to

$$\Psi_k^\dagger = (\psi_{k+Q/2,\uparrow}^\dagger, \psi_{k+Q/2,\downarrow}^\dagger, \psi_{k-Q/2,\uparrow}^\dagger, \psi_{k-Q/2,\downarrow}^\dagger, \psi_{-k-Q/2,\uparrow}, \psi_{-k-Q/2,\downarrow}, \psi_{-k+Q/2,\uparrow}, \psi_{-k+Q/2,\downarrow}) \quad (3)$$

and introduce Pauli matrices  $\rho$  operating in the additional AFM subspace. We thus arrive at  $\mathcal{H} = \frac{1}{2} \sum_k \Psi_k^\dagger [\mathcal{H}^0(k) + \mathcal{V}(k)] \Psi_k$  with

$$\begin{aligned} \mathcal{H}^0(k) &= 2t \cos(Ja/2) \sin(ka) \tau_z \rho_z - M \tau_z \rho_x \sigma_x - \Delta \tau_y \sigma_y, \\ \mathcal{V}(k) &= \mu_B B \tau_z \sigma_z - 2t \sin(Ja/2) \cos(ka) \rho_z. \end{aligned} \quad (4)$$

As a first step we consider  $B = J = 0$ . After performing the unitary transformation  $\Psi_k = U \Psi'_k$  with

$$U = \exp\left(i\frac{\pi}{4} \tau_z \sigma_z\right) \exp\left(i\frac{\pi}{4} \tau_y \rho_y\right) \exp\left(i\frac{\pi}{4} \sigma_x (1 + \tau_z)\right), \quad (5)$$

the Hamiltonian becomes diagonal in the  $\rho$  and  $\sigma$  spaces, yielding the BDI-symmetry-class block Hamiltonian  $\mathcal{H}_{\rho,\sigma}^0(k) = \mathbf{g}_{\rho,\sigma}^0(k) \cdot \boldsymbol{\tau}$  [9, 27] (see also Supl. A), where

$$\mathbf{g}_{\rho,\sigma}^0(k) = (0, (\rho M - \Delta)\sigma, 2t \rho \sin(ka)). \quad (6)$$

The eigenvalues of this Hamiltonian are degenerate in spin space. They are given by

$$E_{\rho,\sigma}^0(k) = \pm \sqrt{(\rho M - \Delta)^2 + 4t^2 \sin^2(ka)}, \quad (7)$$

showing a gap-closing at  $\Delta = M$  for  $\rho = 1$ . In order to discern whether this gap closing leads to a MF zero-mode, which would imply a transition to a TSC, we make use of the specific form of Eq. (6) and introduce the relevant  $\mathbb{Z}$  topological invariant [28] defined by the winding number

$$\tilde{N}_{\rho,\sigma}^0 = \frac{a}{2\pi} \int_{BZ} dk \left( \hat{\mathbf{g}}_{\rho,\sigma}^0(k) \times \frac{\partial \hat{\mathbf{g}}_{\rho,\sigma}^0(k)}{\partial k} \right)_x, \quad (8)$$

with  $\hat{\mathbf{g}}_{\rho,\sigma}^0(k) = \mathbf{g}_{\rho,\sigma}^0(k)/|\mathbf{g}_{\rho,\sigma}^0(k)|$ . It is straightforward to show that the latter is zero, and consequently at this stage, with only AFM order present, there is no transition to a topologically non-trivial SC phase.

Next we add the terms accounting for the control fields  $J$  and  $B$ . We choose  $t$  and  $B$  to be small,  $t, \mu_B B \ll \Delta, M$  and perform a second order expansion based on a canonical transformation  $\tilde{\mathcal{H}} = \mathcal{H}^0 + i/2 [S, \mathcal{V}]$  with  $[S, \mathcal{H}^0] = i\mathcal{V}$  (see Supl. A). The perturbation modifies the energy-spectrum, but only changes in the vicinity of  $k = 0$  lead to a change in the topological properties. Based on this argument, we retain only the most relevant term for small  $k$ , characterized by the coefficient  $\Lambda = 2\mu_B B t \sin(Ja/2)/M$ , and neglect all other second order terms in the expansion parameters  $ka$ ,  $Jat/M$  and  $\mu_B B/M$ . After this expansion we obtain the modified vectors

$$\mathbf{g}_{\rho,\sigma}(k) = (0, (\rho \tilde{M} - \Delta)\sigma + \Lambda \cos(ka), 2\tilde{t} \rho \sin(ka)). \quad (9)$$

The term proportional to  $\Lambda$  lifts the spin-degeneracy in the spectrum, and we obtain

$$E_{\rho,\sigma}(k) = \pm \sqrt{[(\rho \tilde{M} - \Delta)\sigma + \Lambda \cos(ka)]^2 + 4\tilde{t}^2 \sin^2(ka)}.$$

Here we have introduced the renormalized values  $\tilde{M} = M + (\mu_B^2 B^2 + 4t^2 \sin^2(Ja/2))/2M$  and  $\tilde{t} = t \cos(Ja/2)$ . The topological invariant is now given by

$$\tilde{N}_{\rho,\sigma} = \frac{\text{sgn}(\rho \tilde{M} - \Delta + \sigma \Lambda) - \text{sgn}(\rho \tilde{M} - \Delta - \sigma \Lambda)}{2\rho \sigma}, \quad (10)$$

from which we infer that the transition to the topological non-trivial region takes place for  $\tilde{M} + |\Lambda| > \Delta$ . One observes that due to the intrinsic magnetic order there is no requirement for a large external field. In fact, the magnetic field is primarily required for lifting the spin-degeneracy. We also note that the combination of Zeeman field and supercurrent flow gives rise to an effective staggered spin-orbit coupling term (Eq. (11)), akin to the Rashba spin-orbit coupling in TSC nanowires [6, 7].

Based on the bulk-boundary correspondence, we expect that the non-zero topological invariant which we obtain for suitable values of  $J$  and  $B$ , are reflected in the properties of the finite-size system. To demonstrate this correspondence we show now that for the  $\mathbf{g}$  vectors derived above, our model harbors *unpaired* MFs similarly to the situation known from Kitaev's model [25].

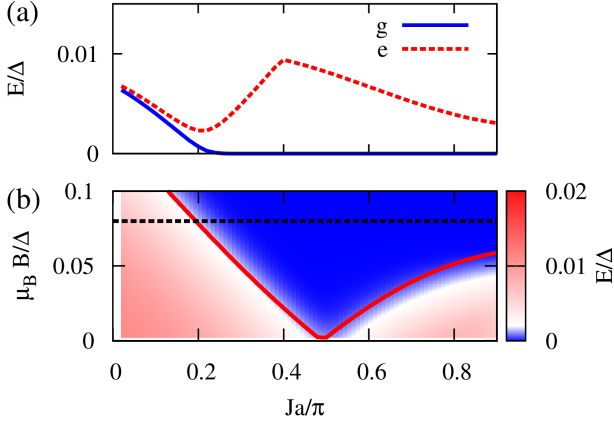


Figure 2: (a) Ground-state (g) and first-excited-state (e) energies, as a function of  $J$  for  $\mu_B B/\Delta = 0.08$  and  $M/\Delta = 0.99$ . (b) Ground-state energy in color scale depending on  $J$  and  $B$ . The red line,  $\tilde{M} + |\Lambda| = \Delta$ , corresponds to the analytically derived critical value for entering the TSC phase, in the *short* superconducting coherence limit. The value  $\mu_B B/\Delta = 0.08$  used in panel (a) is indicated by the black dashed line. ( $N = 160$  lattice sites)

To do so we first transfer to the Majorana basis  $\Psi_k^\dagger \rightarrow \Gamma_n^T = (\gamma_{n\uparrow}, \gamma_{n\downarrow}, \bar{\gamma}_{n\uparrow}, \bar{\gamma}_{n\downarrow})$ , where the superscript  $T$  denotes transposition, and we introduced the MF operators  $\gamma_{n\sigma} \equiv (\psi_{n\sigma} + \psi_{n\sigma}^\dagger)/\sqrt{2}$  and  $\bar{\gamma}_{n\sigma} \equiv (\psi_{n\sigma} - \psi_{n\sigma}^\dagger)/(\sqrt{2}i)$ . In this basis the Hamiltonian reads

$$\begin{aligned} \tilde{\mathcal{H}} = & \frac{1}{2} \sum_{n=1}^N \Gamma_n^T [\Delta \tau_x \sigma_y + (-1)^n \tilde{M} \tau_y \sigma_x] \Gamma_n \\ & + \frac{1}{2} \sum_{n=1}^{N-1} \Gamma_n^T [2\tilde{t} \tau_y - (-1)^n \Lambda \sigma_y] \Gamma_{n+1}. \end{aligned} \quad (11)$$

The first line describes the on-site coupling of MFs. With the objective of unpairing them at each lattice site, we choose  $\Delta = \tilde{M}$ , so that the operators  $\gamma_{n\uparrow}$  and  $\bar{\gamma}_{n\downarrow}$  for odd  $n$  and  $\gamma_{n\downarrow}$  and  $\bar{\gamma}_{n\uparrow}$  for even  $n$  become unpaired. Within this subspace, we choose  $2\tilde{t} = \Lambda$ , and the second line becomes

$$\tilde{\mathcal{H}}_{\text{sub}} = -i\Lambda \sum_{n=1}^{N-1} \gamma_{n,B} \gamma_{n+1,A}. \quad (12)$$

In the last step we introduced the new MF operators:  $\gamma_{2m-1,A} = (\gamma_{2m-1,\uparrow} + \bar{\gamma}_{2m-1,\downarrow})/\sqrt{2}$  and  $\gamma_{2m-1,B} = (\gamma_{2m-1,\uparrow} - \bar{\gamma}_{2m-1,\downarrow})/\sqrt{2}$  as well as  $\gamma_{2m,A} = (\gamma_{2m,\downarrow} + \bar{\gamma}_{2m,\uparrow})/\sqrt{2}$  and  $\gamma_{2m,B} = (\gamma_{2m,\downarrow} - \bar{\gamma}_{2m,\uparrow})/\sqrt{2}$ . The transparent form of the Hamiltonian of Eq. (12) displays directly the *unpaired* MFs  $\gamma_{1,A}$  and  $\gamma_{N,B}$  at the two ends of the chain, as also illustrated in Fig. 1.

Our model analysis can be confirmed by numerically diagonalizing the Hamiltonian built up from the terms of Eqs. (1) and (2). In Fig. 2 we show the lowest positive

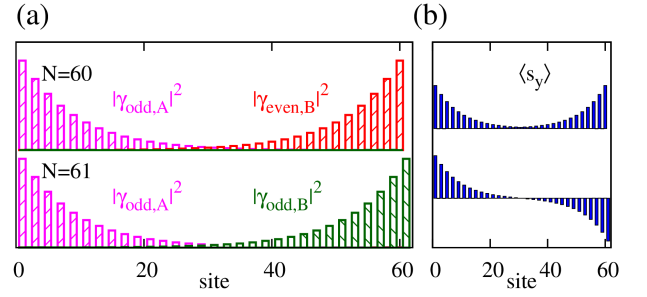


Figure 3: (a) Majorana wave-functions for odd ( $N = 60$ ) and even ( $N = 61$ ) number of sites for the parameters of Fig. 2(a) with  $Ja/\pi = 0.4$ . Depending on the parity of  $N$  the right-edge Majorana wave-function is  $\gamma_{\text{even},B}$  ( $N = 60$ ) or  $\gamma_{\text{odd},B}$  ( $N = 61$ ). (b) corresponding electronic spin-texture.

energies, i.e. the ground state (g) and the first excited state (e). With increasing supercurrent  $J$  the system undergoes a transition to a gapped phase accompanied by a single zero-energy MF solution per edge. This transition occurs close to the previously extracted condition  $\tilde{M} + |\Lambda| = \Delta$  or equivalently  $\mu_B B = |\sqrt{2M|\Delta - M|} - 2t \sin(Ja/2)|$  (see red line in Fig. 2(b)). One observes that the Majorana wave-functions are exponentially suppressed within the bulk. In addition, due to chiral symmetry they are constrained to be zero at every second site (see Supl. B). Depending on the parity of  $N$ , the right-edge Majorana wave-function is either  $\gamma_{\text{even},B}$  or  $\gamma_{\text{odd},B}$ , i.e. it is confined to either even or odd sites. In addition, since the electronic components of the MF wave-functions constitute eigenstates of  $\sigma_y$ ,  $\gamma_{\text{even},B}$  and  $\gamma_{\text{odd},B}$  lead to opposite electronic spin-polarization at the edges, as plotted in Fig. 3(b).

The limit of short coherence length allowed us to transparently expose the underlying TSC mechanism and the related qualitative characteristics of the MFs. However, usually the SC coherence length is rather long, e.g. for Pb it is  $\xi_0 \sim 80$  nm to be compared to the typical spacing of the atoms  $a \sim 1$  nm [23, 24]. We investigate this general case by analyzing a fully *microscopic* model for a chain of AFM ordered magnetic atoms placed on top of a superconductor.

The Hamiltonian of a bulk SC is given by  $\mathcal{H}_S = \sum_{\mathbf{k}} \mathcal{C}_{\mathbf{k}}^\dagger (-\Delta \tau_y \sigma_y + \xi_{\mathbf{k}} \tau_z) \mathcal{C}_{\mathbf{k}}$ , where  $\mathcal{C}_{\mathbf{k}}^\dagger = (c_{\mathbf{k}\uparrow}^\dagger, c_{\mathbf{k}\downarrow}^\dagger, c_{-\mathbf{k}\uparrow}, c_{-\mathbf{k}\downarrow})$  and  $c_{\mathbf{k}\sigma}^\dagger$  creates an electron with momentum  $\mathbf{k}$  and spin  $\sigma$ ,  $\xi_{\mathbf{k}}$  denotes the electronic dispersion, and  $\Delta$  is the energy gap of the bulk superconductor. The resulting quasiparticle excitation spectrum is  $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$ . The presence of magnetic atoms leads to additional states, the so-called Shiba states [29]. They are localized and lie energetically within the energy gap. In the following we treat the magnetic atoms as classical spins characterized by an exchange energy  $M$  with the conduction electrons. Furthermore, we allow for a non-magnetic potential  $U$ . The interaction

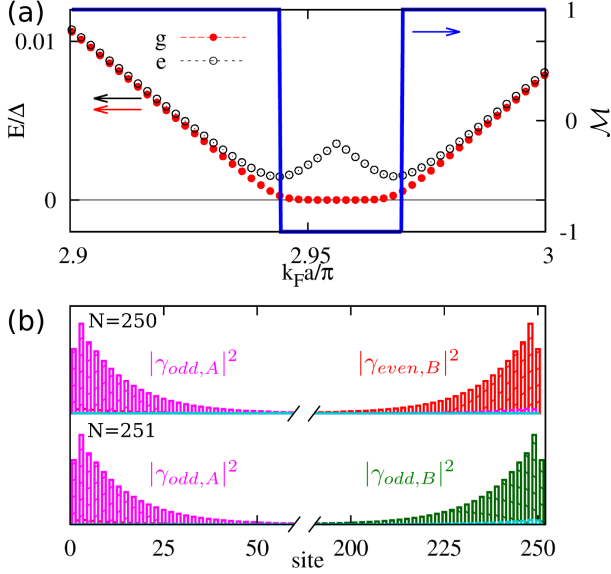


Figure 4: (a) Solution of the microscopic model Eq. (16): Energies of the ground-state (g-red) and excited-state (e-black). The corresponding topological invariant  $\mathcal{M}$  is shown by the blue line (b) Majorana wave-functions in the topologically non-trivial regime.

is given by  $V(\mathbf{r}) = \sum_{n=1}^N \delta(z - na) \delta(x) \delta(y) \mathcal{V}_n$  with  $\mathcal{V}_n = U\tau_z - (-1)^n M\tau_z\sigma_x$ , and the total Hamiltonian can be written as

$$\mathcal{H} = \mathcal{H}_S + \sum_{\mathbf{k}, \mathbf{k}'} \sum_n e^{-i(k_z - k'_z)na} \mathcal{C}_{\mathbf{k}'}^\dagger \mathcal{V}_n \mathcal{C}_{\mathbf{k}}. \quad (13)$$

We solve this problem (see Supl. C) in terms of the Bogoliubov - de Gennes (BdG) equation [18, 30–32],

$$\sum_{\mathbf{k}'} \left[ \delta_{\mathbf{k}, \mathbf{k}'} - G_{\mathbf{k}}(\varepsilon) \sum_n e^{-i(k_z - k'_z)na} \mathcal{V}_n \right] \phi_{\mathbf{k}'} = 0, \quad (14)$$

where  $G_{\mathbf{k}}(\varepsilon) = (\varepsilon + \Delta\tau_y\sigma_y - \xi_{\mathbf{k}}\tau_z)^{-1}$  is the Green's function of the bulk superconductor, and the spinor  $\phi_{\mathbf{k}} = (u_{\mathbf{k}, \uparrow}, u_{\mathbf{k}, \downarrow}, v_{\mathbf{k}, \uparrow}, v_{\mathbf{k}, \downarrow})^T$  contains the spin-dependent particle and hole amplitudes  $u$  and  $v$ . In the presence of a weak in-plane magnetic field and a small supercurrent  $J \ll k_F$  we have to substitute  $\xi_{\mathbf{k}} \rightarrow \xi_{\mathbf{k} - J\hat{z}\tau_z/2} + \mu_B B\sigma_z$  and the linearized Green's function  $G_{\mathbf{k}}$  reads

$$G_{\mathbf{k}}(\varepsilon) \approx \frac{\varepsilon - \Delta\tau_y\sigma_y + \xi_{\mathbf{k}}\tau_z - \mu_B B\tau_z\sigma_z + \frac{\hbar^2 J}{2m} k_F z}{-E_{\mathbf{k}}^2}. \quad (15)$$

As we are interested in low-energy solutions, we performed a linear expansion in  $\varepsilon/\Delta$  as well. In order to further simplify the solution of Eq. (14) we trace out the continuum states, i.e.  $\phi_n = \sum_{\mathbf{k}} e^{ik_z na} \phi_{\mathbf{k}}$  and  $G_n(\varepsilon) = \sum_{\mathbf{k}} e^{ik_z na} G_{\mathbf{k}}(\varepsilon)$ , such that Eq. (14) reduces to

$$\sum_{l=1}^N [\delta_{nl} - G_{n-l}(\varepsilon) \mathcal{V}_l] \phi_l = 0. \quad (16)$$

This equation can be written in the form of a generalized eigenvalue problem  $\sum_{l=1}^N A_{nl} \phi_l = \frac{\varepsilon}{\Delta} \sum_{l=1}^N B_{nl} \phi_l$  which can be readily solved numerically. In momentum space it converts into the form of a usual Schrödinger-equation  $\tilde{\mathcal{H}}_k \psi_k = \varepsilon_k \psi_k$  with spinors  $\psi_k$  related to the original ones by a transformation  $\psi_k = \mathcal{L}_k \phi_k$  and an *effective* Hamiltonian

$$\begin{aligned} \mathcal{H}_k = & -\Delta\tau_y\sigma_y + h_k^{(1)}\tau_z + h_k^{(2)}\tau_z\rho_x\sigma_x + h_k^{(3)}\tau_z\rho_z \\ & + \mu_B B\tau_z\sigma_z + h_k^{(4)} + h_k^{(5)}\rho_z. \end{aligned} \quad (17)$$

The parameters  $h_k^{(i)}$  that are listed in Supl. C. A comparison with Eq. (4) identifies  $h_k^{(2)}$ ,  $h_k^{(3)}$  and  $h_k^{(5)}$  as the magnetic exchange, kinetic energy, and the supercurrent contribution, respectively. In more detail  $h_k^{(2)}$  couples even-order neighbors, while  $h_k^{(3)}$  and  $h_k^{(5)}$  couple odd-order neighbors. The additional terms  $h_k^{(1)}$  and  $h_k^{(4)}$ , correspond to even-order-neighbor kinetic energy and supercurrent contributions, respectively. These parameters crucially depend on how the spacing of the atoms,  $a$ , compares to the coherence length of the superconductor  $\xi_0 \sim \hbar v_F/\Delta$ . In the limit  $a \gg \xi_0$  the microscopically derived Hamiltonian (17) can be expanded in orders of  $\exp(-a/\xi_0)$  leading to Eq. (4) with  $t \sim \exp(-a/\xi_0)$ .

The microscopic Hamiltonian belongs to symmetry class D and its topological properties can be investigated using the  $\mathbb{Z}_2$  topological invariant  $\mathcal{M}$  [25], defined as  $\mathcal{M} \equiv \text{sgn} \{ \text{Pf}[k=0] \text{Pf}[k=\pi/2a] \}$ . Note that the second Pfaffian is evaluated at  $k = \pi/2a$ , since due to the AFM we have to consider the folded Brillouin zone. In the parameter-space considered here, only the  $k = 0$  component changes sign and we obtain

$$\mathcal{M} = \text{sgn} [(\mu_B B)^4 - 2(\mu_B B)^2(A^2 + h_5^2) + (A^2 - h_5^2)^2] \quad (18)$$

Here we used the short-hand notation  $h_j = h_{k=0}^{(j)}$  and  $A^2 = \Delta^2 + h_1^2 - h_2^2$ . The system turns topologically non-trivial when  $\mathcal{M}$  changes sign, which occurs when  $\mu_B B = |A - h_5|$ . This condition is equivalent to the one we found for the minimal model.

For the numerical analysis we choose  $\xi_0/a = 20\pi$ , a magnetic field  $\mu_B B/\Delta = 0.05$  which for Pb would be of the order  $B \sim 1\text{T}$ , and a supercurrent  $J$  such that  $\hbar^2 J k_F / 2m\Delta = 0.5$  ( $k_F$  is the Fermi-momentum and  $J \sim 0.1 J_c$ , where  $J_c \simeq 1/\xi_0$  is the critical current of the superconductor). As shown in Fig. 4(a)  $\mathcal{M}$  changes sign as a function of the atomic spacing  $a$  in a window where the system exhibits a gap between zero energy state and first excited state. Note that the window of the TSC phase can be further broadened by increasing the supercurrent flowing through the substrate. Furthermore, for this region the ground state wave-functions, presented in Fig. 4(b), feature the *same* characteristics previously obtained within our minimal model.

In conclusion we propose a new hybrid system based on an array of antiferromagnetically aligned magnetic

atoms in proximity to a conventional superconductor which allows engineering edge Majorana fermions. Such antiferromagnetically ordered atomic chains on metallic substrates have already been fabricated and manipulated. Furthermore, STM techniques can be used for performing spin-polarized zero-bias anomaly spectroscopy which can resolve the electronic spin texture of the Majorana wave-functions. In fact, the edge spin-polarization can be reversed by changing the length of the chain by a *single* atom. The latter property is robust and can identify the emergence of MFs. Furthermore, our device can be finely controlled by the combination of supercurrents and *weak* magnetic fields, offering a rich test ground for experiments.

The enhanced functionality can be also attractive for applications of Majorana fermions for quantum information processing. Finally, our proposal can be further extended to a setup where the AFM order is established by nano-magnets [11] or to materials which already exhibit a microscopic coexistence of superconductivity and intrinsic antiferromagnetism such as Fe-[33] or Ce-[34] based superconductors.

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## Supplementary material

### A. Schrieffer - Wolff transformation

We perform a perturbative expansion of the Hamiltonian  $\mathcal{H} = \mathcal{H}^0 + \mathcal{V}$  of Eq. (4), where  $\mathcal{H}_0 = \frac{1}{2} \sum_k \Gamma_{-k}^T \mathcal{H}_k^0 \Gamma_k$  and  $\mathcal{V} = \frac{1}{2} \sum_k \Gamma_{-k}^T \mathcal{V}_k \Gamma_k$  with  $\mathcal{H}_k^0 = \Delta \tau_x \sigma_y + M \tau_y \rho_x \sigma_x - 2t \cos(Ja/2) \sin(ka) \tau_y \rho_z$  and  $\mathcal{V}_k = -\mu_B B \tau_y \sigma_z - 2t \sin(Ja/2) \cos(ka) \rho_z$ . To obtain the above expressions, we first introduced  $\gamma_{k\sigma} = (\psi_{k\sigma} + \psi_{-k\sigma}^\dagger)/\sqrt{2}$  and  $\bar{\gamma}_{k\sigma} = (\psi_{k\sigma} - \psi_{-k\sigma}^\dagger)/(\sqrt{2}i)$  and then the Majorana-spinors  $\Gamma_k^T = (\gamma_{k\uparrow}, \gamma_{k\downarrow}, \bar{\gamma}_{k\uparrow}, \bar{\gamma}_{k\downarrow})$ . Following Schrieffer and Wolff we are interested in a unitary transformation  $\mathcal{U} = e^{i\mathcal{S}}$  with  $\mathcal{S} = \mathcal{S}^\dagger$  so that the linear order in  $\mathcal{V}$  is eliminated in  $\tilde{\mathcal{H}} = \mathcal{U}\mathcal{H}\mathcal{U}^\dagger$ . The expansion can be expressed as follows  $\tilde{\mathcal{H}} = \mathcal{H} + i[\mathcal{S}, \mathcal{H}] - \frac{1}{2}[\mathcal{S}, [\mathcal{S}, \mathcal{H}]] + \dots$ . The operator  $\mathcal{S}$  is chosen such that  $[\mathcal{S}, \mathcal{H}^0] = i\mathcal{V}$  so that up to second order in  $\mathcal{V}$  the Hamiltonian reads  $\tilde{\mathcal{H}} = \mathcal{H}^0 + i[\mathcal{S}, \mathcal{V}]/2$ . Further parametrizing  $\mathcal{S}$  by  $\mathcal{S} = \frac{1}{2} \sum_k \Gamma_{-k}^T \mathcal{S}_k \Gamma_k$  we find that

$$\mathcal{S}_k = \frac{t \sin(Ja/2) \cos(ka)}{M} \tau_y \rho_y \sigma_x - \frac{t^2 \sin(Ja) \sin(2ka)}{2M\Delta} \tau_x \rho_x \sigma_z + \frac{\mu_B B}{2M} \rho_x \sigma_y - \frac{\mu_B B t \cos(Ja/2) \sin(ka)}{M\Delta} \tau_z \rho_y \quad (19)$$

and with this that the transformed Hamiltonian is given by

$$\tilde{\mathcal{H}}_k = \Delta \tau_x \sigma_y + \left( M + \frac{[\mu_B B]^2 + 4t^2 \sin^2(Ja/2) \cos^2(ka)}{2M} \right) \tau_y \rho_x \sigma_x - 2t \cos(Ja/2) \sin(ka) \tau_y \rho_z \quad (20)$$

$$\begin{aligned} & - \frac{\mu_B B t^2 \sin(Ja) \sin(2ka)}{\Delta M} \tau_z \rho_x - \frac{2\mu_B B t \sin(Ja/2) \cos(ka)}{M} \rho_y \sigma_y \\ & + \frac{t \cos(Ja/2) \sin(ka) \{ (\mu_B B)^2 + [2t \sin(Ja/2) \cos(ka)]^2 \}}{\Delta M} \tau_x \rho_y \sigma_z. \end{aligned} \quad (21)$$

The topological properties of the system are determined by changes at the points  $k = 0, \pi/(2a)$ . Neglecting all terms which are linear in  $k$  close to these points and at least second order in the expansion parameters  $t, \mu_B B \ll M, \Delta$  we obtain the effective low energy Hamiltonian

$$\begin{aligned} \tilde{\mathcal{H}}_k \approx & \Delta \tau_x \sigma_y + \left( M + \frac{[\mu_B B]^2 + 4t^2 \sin^2(Ja/2) \cos^2(ka)}{2M} \right) \tau_y \rho_x \sigma_x \\ & - 2t \cos(Ja/2) \sin(ka) \tau_y \rho_z - \frac{2\mu_B B t \sin(Ja/2) \cos(ka)}{M} \rho_y \sigma_y, \end{aligned} \quad (22)$$

which converts to Eq. (11) by transferring to the lattice representation. After suitably rewriting the above Hamiltonian in the original spinor  $\Psi_k$  space, we observe that it assumes a generalized time-reversal symmetry  $\Theta = i\rho_y \sigma_y \mathcal{K}'$ , i.e.  $[\Theta, \tilde{\mathcal{H}}_k] = 0$ , together with a charge-conjugation symmetry  $\Xi = \tau_x \rho_x \mathcal{K}'$ , i.e.  $\{\Xi, \tilde{\mathcal{H}}_k\} = 0$ . Here  $\mathcal{K}'$  defines complex conjugation, where the prime indicates that it does not act on  $Q$ . With this, also the chiral operator  $\Pi = \Theta \Xi = \tau_x \rho_z \sigma_y$  anticommutes with the Hamiltonian, i.e.  $\{\Pi, \tilde{\mathcal{H}}_k\} = 0$ . Evenmore, there exist two unitary symmetries  $[\tau_y \rho_y, \tilde{\mathcal{H}}_k] = [\tau_z \rho_z \sigma_x, \tilde{\mathcal{H}}_k] = 0$ , yielding the additional time-reversal, charge conjugation and chiral symmetries  $[\tau_y \rho_y \Theta, \tilde{\mathcal{H}}_k] = [\tau_z \rho_z \sigma_x \Theta, \tilde{\mathcal{H}}_k] = 0$ ,  $\{\tau_y \rho_y \Xi, \tilde{\mathcal{H}}_k\} = \{\tau_z \rho_z \sigma_x \Xi, \tilde{\mathcal{H}}_k\} = 0$  and  $\{\tau_y \rho_y \Pi, \tilde{\mathcal{H}}_k\} = \{\tau_z \rho_z \sigma_x \Pi, \tilde{\mathcal{H}}_k\} = 0$ , respectively. Since  $\Theta^2 = \Xi^2 = \Pi^2 = 1$  the Hamiltonian belongs to the symmetry class  $\oplus_{n=1}^4 \text{BDI}$  [9].

### B. Kitaev Chain

In this section we discuss the mapping to the Kitaev chain starting with the Hamiltonian in Eq. (11),

$$\begin{aligned} \tilde{\mathcal{H}} = & -\frac{i}{2} \sum_{n=1}^N \begin{pmatrix} \gamma_{n\uparrow} \\ \gamma_{n\downarrow} \\ \bar{\gamma}_{n\uparrow} \\ \bar{\gamma}_{n\downarrow} \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & \Delta + (-1)^n \tilde{M} \\ 0 & 0 & -\Delta + (-1)^n \tilde{M} & 0 \\ 0 & \Delta - (-1)^n \tilde{M} & 0 & 0 \\ -\Delta - (-1)^n \tilde{M} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_{n\uparrow} \\ \gamma_{n\downarrow} \\ \bar{\gamma}_{n\uparrow} \\ \bar{\gamma}_{n\downarrow} \end{pmatrix} \\ & + \frac{i}{2} \sum_{n=1}^{N-1} \begin{pmatrix} \gamma_{n\uparrow} \\ \gamma_{n\downarrow} \\ \bar{\gamma}_{n\uparrow} \\ \bar{\gamma}_{n\downarrow} \end{pmatrix}^T \begin{pmatrix} 0 & (-1)^n \Lambda & -2\tilde{t} & 0 \\ -(-1)^n \Lambda & 0 & 0 & -2\tilde{t} \\ 2\tilde{t} & 0 & 0 & (-1)^n \Lambda \\ 0 & 2\tilde{t} & -(-1)^n \Lambda & 0 \end{pmatrix} \begin{pmatrix} \gamma_{n+1\uparrow} \\ \gamma_{n+1\downarrow} \\ \bar{\gamma}_{n+1\uparrow} \\ \bar{\gamma}_{n+1\downarrow} \end{pmatrix}. \end{aligned} \quad (23)$$

Choosing  $\Delta = \widetilde{M}$  we find that  $\Delta + (-1)^n \widetilde{M} = 0$  for odd  $n$  and  $\Delta - (-1)^n \widetilde{M} = 0$  for even  $n$ , meaning that the on-site coupling, for instance between  $\gamma_{n\uparrow}$  and  $\bar{\gamma}_{n\downarrow}$ , is cancelled at odd sites  $n$ . The same is true for  $\gamma_{n\downarrow}$  and  $\bar{\gamma}_{n\uparrow}$  at even sites  $n$ . Because of that the nearest-neighbor-coupling in the second line of Eq. (23) can be fully expressed within the subspace  $\{\gamma_{\text{odd},\uparrow}, \bar{\gamma}_{\text{odd},\downarrow}, \gamma_{\text{even},\downarrow}, \bar{\gamma}_{\text{even},\uparrow}\}$ , i.e.

$$\widetilde{\mathcal{H}}_{\text{sub}} = \frac{i}{2} \sum_m \begin{pmatrix} \gamma_{2m-1\uparrow} \\ \bar{\gamma}_{2m-1\downarrow} \end{pmatrix}^T \begin{pmatrix} -\Lambda & -2\widetilde{t} \\ 2\widetilde{t} & \Lambda \end{pmatrix} \begin{pmatrix} \gamma_{2m\downarrow} \\ \bar{\gamma}_{2m\uparrow} \end{pmatrix} + \frac{i}{2} \sum_m \begin{pmatrix} \gamma_{2m\downarrow} \\ \bar{\gamma}_{2m\uparrow} \end{pmatrix}^T \begin{pmatrix} -\Lambda & -2\widetilde{t} \\ 2\widetilde{t} & \Lambda \end{pmatrix} \begin{pmatrix} \gamma_{2m+1\uparrow} \\ \bar{\gamma}_{2m+1\downarrow} \end{pmatrix}. \quad (24)$$

Especially for  $\Lambda = 2\widetilde{t}$  one finds that

$$\widetilde{\mathcal{H}}_{\text{sub}} = \frac{i}{2} \Lambda \sum_m \begin{pmatrix} \gamma_{2m-1\uparrow} \\ \bar{\gamma}_{2m-1\downarrow} \end{pmatrix}^T \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{2m\downarrow} \\ \bar{\gamma}_{2m\uparrow} \end{pmatrix} + \frac{i}{2} \Lambda \sum_m \begin{pmatrix} \gamma_{2m\downarrow} \\ \bar{\gamma}_{2m\uparrow} \end{pmatrix}^T \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{2m+1\uparrow} \\ \bar{\gamma}_{2m+1\downarrow} \end{pmatrix} \quad (25)$$

$$= -i\Lambda \sum_m \left( \frac{\gamma_{2m-1\uparrow} - \bar{\gamma}_{2m-1\downarrow}}{\sqrt{2}} \frac{\gamma_{2m\downarrow} + \bar{\gamma}_{2m\uparrow}}{\sqrt{2}} + \frac{\gamma_{2m\downarrow} - \bar{\gamma}_{2m\uparrow}}{\sqrt{2}} \frac{\gamma_{2m+1\uparrow} + \bar{\gamma}_{2m+1\downarrow}}{\sqrt{2}} \right) = -i\Lambda \sum_{n=1}^{N-1} \gamma_{n,B} \gamma_{n+1,A} \quad (26)$$

Like in the main text we have introduced the new Majorana-operators  $\gamma_{2m-1,A} = (\gamma_{2m-1,\uparrow} + \bar{\gamma}_{2m-1,\downarrow})/\sqrt{2}$ ,  $\gamma_{2m-1,B} = (\gamma_{2m-1,\uparrow} - \bar{\gamma}_{2m-1,\downarrow})/\sqrt{2}$ ,  $\gamma_{2m,A} = (\gamma_{2m,\downarrow} + \bar{\gamma}_{2m,\uparrow})/\sqrt{2}$  and  $\gamma_{2m,B} = (\gamma_{2m,\downarrow} - \bar{\gamma}_{2m,\uparrow})/\sqrt{2}$ . It turns out that the electronic part  $|\gamma_{n,A/B}\rangle_{\text{el}}$  of the latter Majorana wave-functions are eigenstates of the spin-operator  $s_y = \hbar\sigma_y/2$ . We can see this by going back to the representation in terms of the original fermion operators – for instance  $\gamma_{\text{odd},A} = (\psi_{\text{odd},\uparrow} - i\psi_{\text{odd},\downarrow}) + (\psi_{\text{odd},\uparrow}^\dagger + i\psi_{\text{odd},\downarrow}^\dagger) = \psi_{\text{odd},\rightarrow} + \psi_{\text{odd},\rightarrow}^\dagger$ . With the new fermion operators  $\psi_{\text{odd},\rightleftharpoons} = \psi_{\text{odd},\uparrow\downarrow} - i\psi_{\text{odd},\downarrow\uparrow}$  and  $|\gamma_A\rangle_{\text{el}} \equiv \psi_{\rightleftharpoons}^\dagger|0\rangle$  being an eigenket of  $s_y = \hbar\sigma_y/2$  with eigenvalue  $\pm\hbar/2$ . With this we find that

$${}_{\text{el}}\langle\gamma_{n,A}|s_y|\gamma_{n,A}\rangle_{\text{el}} = -(-1)^n \frac{\hbar}{2} \quad \text{and} \quad {}_{\text{el}}\langle\gamma_{n,B}|s_y|\gamma_{n,B}\rangle_{\text{el}} = (-1)^n \frac{\hbar}{2}. \quad (27)$$

According to the symmetry analysis in Sup. A the Majorana wave-functions constitute eigen-vectors of the chiral symmetry operators  $\Pi_{\mathcal{M}} = \tau_z \rho_z \sigma_y = i\tau_z \sigma_y t_{\pi/Q}$ ,  $\Pi'_{\mathcal{M}} = \tau_y \rho_x \sigma_y$  and  $\Pi''_{\mathcal{M}} = \tau_x \sigma_z$  (expressed now in the MF basis). Here  $t_{\pi/Q} \psi_n = \psi_{n+1}$  is the translation operator which in momentum space – up to a phase – corresponds to  $t_{\pi/Q} = -i\rho_z$ . The eigenstates of  $t_{\pi/Q}$  have the property  $t_{\pi/Q} \psi_n = \psi_{n+1} = \lambda \psi_n$ . At the same time  $t_{2\pi/Q} = t_{\pi/Q}^2$  commutes with the Hamiltonian such that  $t_{2\pi/Q} \psi_n = \psi_{n+2} = \lambda^2 \psi_n$  and therefore  $\lambda = \pm 1$ . This means that eigenfunctions  $\psi_n^{(\pm)}$  of  $t_{\pi/Q}$  are either constant or alternating at each lattice site, i.e.  $\psi_n^{(+)} = 1$  or  $\psi_n^{(-)} = (-1)^n$ . Accordingly, the  $\rho_x$  operator has eigen-functions  $\psi_n = 1 \pm (-1)^n$ , this means they vanish every second site. Since the related Majorana wave-function  $A$  or  $B$  constitutes an eigen-state of  $\tau_y \sigma_y$ , they must be also eigen-states of  $\rho_x$  and demonstrate this even-odd property. This can be seen for instance in Fig. 3.

### C. Bogoliubov - de Gennes equation

Starting with the Green's function  $G_{\mathbf{k}}(\varepsilon)$  in Eq. (15) we retrieve the Fourier-components  $G_n(\varepsilon) = \sum_{\mathbf{k}} e^{ik_z n a} G_{\mathbf{k}}(\varepsilon)$  in Eq. (16) in terms of a quasiclassical expansion  $\xi_{\mathbf{k}}/\hbar v_F \ll k_F$ , where  $k_F$  and  $v_F$  are the Fermi wave-vector and velocity [31]. To this end the momentum summation is converted to an integral over the linearized dispersion  $\xi$  first,

$$\begin{aligned} G_n(\varepsilon) &= \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_n} \hat{G}_{\mathbf{k}}(\varepsilon) = \mathcal{N}_F \int_{-1}^1 \frac{du}{2} \int_{-\infty}^{\infty} d\xi e^{ik_F u n a} e^{i\frac{\xi}{\hbar v_F} u n a} G_{\mathbf{k}}(\varepsilon) \\ &\approx \mathcal{N}_F \int_{-\infty}^{\infty} d\xi \int_{-1}^1 \frac{du}{2} e^{ik_F u n a} e^{i\frac{\xi}{\hbar v_F} u n a} \frac{\varepsilon + \frac{\hbar^2 J k_F u}{2m} + \xi \tau_z - \Delta \tau_y \sigma_y - \mu_B B \tau_z \sigma_z}{-\xi^2 - \Delta^2} \end{aligned} \quad (28)$$

Here the normal density of states of the superconductor is denoted by  $\mathcal{N}_F$ . The integral over  $u = \cos(\theta)$  (with  $\theta$  being the angle between  $\mathbf{r}_n$  and  $\mathbf{k}$ ) gives

$$\begin{aligned} G_n(\varepsilon) &= \mathcal{N}_F \int d\xi \frac{\sin(k_F n a + \frac{\xi}{\hbar v_F} n a)}{k_F n a + \frac{\xi}{\hbar v_F} n a} \frac{\varepsilon + \xi \tau_z - \Delta \tau_y \sigma_y - \mu_B B \tau_z \sigma_z}{-\xi^2 - \Delta^2} \\ &\quad - i\mathcal{N}_F \int d\xi \left[ \frac{\sin(k_F n a + \frac{\xi}{\hbar v_F} n a) - (k_F n a + \frac{\xi}{\hbar v_F} n a) \cos(k_F n a + \frac{\xi}{\hbar v_F} n a)}{(k_F n a + \frac{\xi}{\hbar v_F} n a)^2} \right] \frac{\hbar^2 J k_F}{2m(\xi^2 + \Delta^2)}. \end{aligned} \quad (29)$$

Furthermore performing the quasiclassical expansion in  $\xi/\hbar v_F \ll k_F$  leads to

$$G_n(\varepsilon) \approx -\pi \mathcal{N}_F e^{-\frac{a|n|}{\xi_0}} \left[ \left( \frac{\varepsilon}{\Delta} - \tau_y \sigma_y - \frac{\mu_B B}{\Delta} \tau_z \sigma_z \right) \frac{\sin(k_F a |n|)}{k_F a |n|} + \tau_z \frac{\cos(k_F a |n|)}{k_F a |n|} + i \frac{\hbar^2 J k_F}{2m\Delta} \text{sgn}(n) \frac{\sin(k_F a |n|) - k_F a |n| \cos(k_F a |n|)}{(k_F a n)^2} \right]. \quad (30)$$

This expression is valid for  $n \neq 0$  and has to be replaced by  $G_n(\varepsilon) = -\pi \mathcal{N}_F (\varepsilon/\Delta - \tau_y \sigma_y - \mu_B B/\Delta \tau_z \sigma_z)$  for  $n = 0$ . We find that the coupling between the spins crucially depends on the coherence length of the superconductor,  $\xi_0 = \hbar v_F/\Delta$ , which has to be distinguished from the dispersion-energy  $\xi$  used in the equations above. In Eq. (16) it enters by the exponential factor  $\exp(-|n-l|a/\xi_0)$ . This means that for  $\xi_0 \lesssim a$  coupling is reduced only to a few neighbors, whereas for  $\xi_0 > a$  it is extended over many.

At this point it is convenient to write Eq. (16) in terms of a generalized eigenvalue-problem,

$$\sum_{l=1}^N A_{nl} \phi_l = \frac{\varepsilon}{\Delta} \sum_{l=1}^N B_{nl} \phi_l, \quad (31)$$

with the combinations

$$A_{nl} = \delta_{nl} \mathcal{V}_l + \pi \mathcal{N}_F \mathcal{V}_n \left[ -(\tau_y \sigma_y + \mu_B B/\Delta \tau_z \sigma_z) a_{n-l} + \tau_z b_{n-l} + c_{n-l} \right] \mathcal{V}_l \quad \text{and} \quad B_{nl} = -\pi \mathcal{N}_F \mathcal{V}_n a_{n-l} \mathcal{V}_l, \quad (32)$$

while

$$\begin{aligned} a_{n-l} &= \sin(k_F a |n-l|) e^{-a|n-l|/\xi_0} / k_F a |n-l|, \\ b_{n-l} &= \cos(k_F a |n-l|) e^{-a|n-l|/\xi_0} / k_F a |n-l|, \\ c_{n-l} &= i \frac{\hbar^2 J k_F}{2m\Delta} \text{sgn}(n-l) e^{-\frac{a|n-l|}{\xi_0}} \frac{\sin(k_F a |n-l|) - k_F a |n-l| \cos(k_F a |n-l|)}{(k_F a |n-l|)^2}. \end{aligned}$$

Because the parameters  $a_{n-l}$ ,  $b_{n-l}$  and  $c_{n-l}$  only depend on the difference  $n-l$  the eigenvalue equation (16) assumes a diagonal form in momentum space. Mind that  $\mathcal{V}_n = U \tau_z - (-1)^n M \tau_z \sigma_x = U \tau_z + \exp(iQna) M \tau_z \sigma_x$ , where  $Q = \pi/a$  is the so called antiferromagnetic wave-vector. Respectively we have to enlarge the spinor by this additional antiferromagnetic subspace in momentum space

$$\phi_k = \left( u_{k+Q/2,\uparrow} \ u_{k+Q/2,\downarrow} \ u_{k-Q/2,\uparrow} \ u_{k-Q/2,\downarrow} \ v_{k+Q/2,\uparrow} \ v_{k+Q/2,\downarrow} \ v_{k-Q/2,\uparrow} \ v_{k-Q/2,\downarrow} \right)^T. \quad (33)$$

Furthermore by introducing  $\mathcal{V} = U \tau_z - M \tau_z \rho_x \sigma_z$  and the Fourier components

$$\begin{aligned} \alpha_k &= \sum_n e^{-ikan} (-i\rho_z)^n a_n = \alpha_k^{(e)} + \alpha_k^{(o)} \rho_z, \\ \beta_k &= \sum_n e^{-ikan} (-i\rho_z)^n b_n = \beta_k^{(e)} + \beta_k^{(o)} \rho_z, \\ \gamma_k &= -i \sum_n e^{-ikan} (-i\rho_z)^n c_n = \gamma_k^{(e)} + \gamma_k^{(o)} \rho_z, \end{aligned} \quad (34)$$

we obtain

$$\mathcal{A}_k \phi_k = \frac{\varepsilon_k}{\Delta} \mathcal{B}_k \phi_k \quad (35)$$

with  $\mathcal{A}_k = \mathcal{V} - \pi \mathcal{N}_F \mathcal{V} [\alpha_k (\tau_y \sigma_y + \mu_B B/\Delta \tau_z \sigma_z) - \beta_k \tau_z - \gamma_k] \mathcal{V}$  and  $\mathcal{B}_k = -\pi \mathcal{N}_F \mathcal{V} \alpha_k \mathcal{V}$ . As in the main text the Pauli-matrices  $\rho_i$  account for the additional AFM subspace. A Cholesky decomposition of the right-hand-side, i.e.  $\mathcal{B}_k = (\mathcal{L}_k \mathcal{L}_k^\dagger)^{-1}/\Delta$ , together with the transformation  $\psi_k = \mathcal{L}_k \phi_k$  leads to the effective Schrödinger-equation

$$[\mathcal{L}_k^\dagger \mathcal{A}_k \mathcal{L}_k] \psi_k = [-\Delta \tau_y \sigma_y + h_k^{(1)} \tau_z + h_k^{(2)} \tau_z \rho_x \sigma_x + h_k^{(3)} \tau_z \rho_z + \mu_B B \tau_z \sigma_z + h_k^{(4)} + h_k^{(5)} \rho_z] \psi_k = \varepsilon_k \psi_k. \quad (36)$$



The respective components of the Hamiltonian are given by

$$\begin{aligned}
\frac{h_k^{(1)}}{\Delta} &= \frac{U\alpha_k^{(e)}}{\pi\mathcal{N}_F(M^2 - U^2)D_k} - \frac{\alpha_k^{(e)}\beta_k^{(e)} - \alpha_k^{(o)}\beta_k^{(o)}}{D_k}, \\
\frac{h_k^{(2)}}{\Delta} &= \frac{-M}{\pi\mathcal{N}_F(M^2 - U^2)\sqrt{D_k}}, \\
\frac{h_k^{(3)}}{\Delta} &= \frac{-U\alpha_k^{(o)}}{\pi\mathcal{N}_F(M^2 - U^2)D_k} - \frac{\alpha_k^{(e)}\beta_k^{(o)} - \alpha_k^{(o)}\beta_k^{(e)}}{D_k}, \\
\frac{h_k^{(4)}}{\Delta} &= -\frac{\alpha_k^{(e)}\gamma_k^{(e)} - \alpha_k^{(o)}\gamma_k^{(o)}}{D_k}, \\
\frac{h_k^{(5)}}{\Delta} &= -\frac{\alpha_k^{(e)}\gamma_k^{(o)} - \alpha_k^{(o)}\gamma_k^{(e)}}{D_k}.
\end{aligned} \tag{37}$$

with the denominator  $D_k = ([\alpha_k^{(e)}]^2 - [\alpha_k^{(o)}]^2)$ . The system now crucially depends on the spacing,  $a$ , of the atoms as well, which affects the parameters  $a_n$ ,  $b_n$  and  $c_n$  in Eq. (32). Note that for short coherence lengths  $\xi_0 \ll a$  the components (34) can be expanded in orders of  $\exp(-a/\xi_0)$ , i.e.  $\alpha_k = a_0 + 2a_1 \sin(ka)\rho_z$ ,  $\beta_k = 2b_1 \sin(ka)\rho_z$  and  $\gamma_k = 2c_1 \cos(ka)\rho_z$ . In this limit setting  $U = 0$  the Hamiltonian (17) becomes equivalent to Eq. (4) in the main text. Within this expansion the components  $h_k^{(1)}$ ,  $h_k^{(2)}$  and  $h_k^{(5)}$ , which enter the topological invariant in Eq. (18), are given by

$$\frac{h_{k=0}^{(1)}}{\Delta} \approx 0, \quad \frac{h_{k=0}^{(2)}}{\Delta} \approx -\frac{1}{\pi\mathcal{N}_F M} \quad \text{and} \quad \frac{h_{k=0}^{(5)}}{\Delta} \approx \frac{\hbar^2 k_F J}{m\Delta} \exp(-a/\xi_0) \frac{\sin(k_F a) - k_F a \cos(k_F a)}{(k_F a)^2}. \tag{38}$$

Substituting  $M' = (1/\pi\mathcal{N}_F M)$  and  $J' = h_{k=0}^{(5)}$  we obtain the condition for the topological phase-transition, i.e.  $\mu_B B = |\sqrt{(\Delta + M')(\Delta - M')} - J'|$  which in the limit  $M' \sim \Delta$  is equivalent to the condition we obtained for the minimal model.

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